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Let's twist again: general metrics of G_2 holonomy from gauged supergravity

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Abstract: We construct all complete metrics of cohomogeneity one G_2 holonomy with $S^3 \times S^3$ principal orbits from gauged supergravity. Our approach rests on a generalization of the twisting procedure used in this framework. It corresponds to a non-trivial embedding of the special lagrangian three-cycle wrapped by the D6-branes in the lower dimensional supergravity. There are constraints that neatly reduce the general ansatz to a six functions one. Within this approach, the Hitchin system and the flop transformation are nicely realized in eight dimensional gauged supergravity.

Keywords: M-Theory, D-branes, String Duality, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

Four dimensional supersymmetric Yang-Mills theory arises in M-theory on a manifold X with G_2 holonomy. If the manifold is large enough and smooth, the low energy description is given in terms of a purely gravitational configuration of eleven dimensional supergravity. The gravity/gauge theory correspondence then allows for a geometrical approach to the study of important aspects of the strong coupling regime of supersymmetric Yang-Mills theory such as the existence of a mass gap [1, 2], chiral symmetry breaking [2], confinement [3], gluino condensation [2, 4], domain walls [5] and chiral fermions [6]. These facts led, in the last two years, to a concrete and important physical motivation to study compact and non-compact seven-manifolds of G_2 holonomy.

Up to last year, there were only three known examples of complete metrics with G_2 holonomy on riemannian manifolds [7, 8]. They correspond to $\mathbb{R}^3$ bundles over S^4 or $\mathbb{CP}^2$, and to an $\mathbb{R}^4$ bundle over S^3. These manifolds develop isolated conical singularities corresponding, respectively, to cones on $\mathbb{CP}^3$, $\mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$, or $S^3 \times S^3$, and the

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dynamics of M-theory on them has been recently studied in great detail [9]. See also [10]. In the last case, in particular, it was shown that there is a moduli space with three branches, and the quotient by a finite subgroup of SU(2) leads either to the uplift of D6-branes wrapping a special lagrangian S^3 in a Calabi-Yau three-fold, or to a smooth manifold admitting no normalizable supergravity zero modes. M-theory on the latter has no massless fields localized in the transverse four-dimensional spacetime. By a smooth interpolation between these manifolds, M-theory realizes the mass gap of $\mathcal{N} = 1$ supersymmetric four-dimensional gauge theory [1, 2]; this geometric dual description corresponding, however, to type-IIA strings at infinite coupling.

We will concentrate on this paper in the case of an $\mathbb{R}^4$ bundle over S^3. Supersymmetry and holonomy matching indicate that a large class of G_2 holonomy manifolds, describing the M-theory lift of D6-branes wrapping a special lagrangian three-cycle on a Calabi-Yau three-fold, must exist [11]. Constructing their complete metrics is an important issue in improving our understanding of the strongly coupled infrared dynamics of $\mathcal{N} = 1$ supersymmetric gauge theories. For example, a new G_2 holonomy manifold with an asymptotically stabilized S^1 — thus describing the M-theory lift of wrapped D6-branes, mentioned in the previous paragraph, in the case of finite string coupling — was recently found [12]. This solution is asymptotically locally conical (ALC) — near infinity it approaches a circle bundle with fibres of constant length over a six-dimensional cone, — as opposed to the asymptotically conical (AC) solutions found in [7, 8].

There have been some attempts of a generic approach to build this kind of complete and non-singular metrics of G_2 holonomy. A rather general system of first-order equations for the metric was obtained in [13, 14] from the BPS domain wall equations corresponding to an auxiliary superpotential. Three types of regular metrics were shown to arise from this system, in which the orbits degenerate respectively to S^3 [12], $T^{1,1}$ [14] and S^2 [15]. With the notable exception of the first one, the solutions are only known numerically. A different fairly general technique was used in [16] by directly exploiting the fact that G_2 holonomy metrics are determined by a three-form that is closed and co-closed. Following a different approach, Hitchin gave a prescription dealing with a hamiltonian system, which is obtained by extremising diffeomorphism invariant functionals on certain differential forms, that leads to metrics of G_2 holonomy [17]. This procedure was then exploited [18] to obtain a general system of first-order equations for metrics of G_2 holonomy with $S^3 \times S^3$ principal orbits that was shown to encompass the previous ones. It was also shown in [18] that, through different contraction limits, G_2 metrics with $S^3 \times T^3$ principal orbits can be attained [19].

On the other hand, a more systematical approach, started in [20], explicitly exploits the fact that these metrics come from the uplift of D6-branes wrapping special lagrangian three-cycles on a Calabi-Yau three-fold. The key issue is given by the non-trivial geometry of the world-volume that forces supersymmetry to be appropriately twisted such that covariantly constant Killing spinors are supported [21]. A natural framework to perform the above mentioned twisting is given by lower dimensional gauged supergravities, whose domain wall solutions usually correspond to the near horizon limit of D-brane configurations [22] thus giving directly the gravity dual description of the gauge theories living on their world-volumes. This approach has been largely followed throughout the literature on

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the subject [23]–[33]. In particular, a generic approach to obtain G_2 holonomy manifolds from eight-dimensional gauged supergravity was undertaken in [28], where it was shown that the conventional twisting prescription should be generalized in a way that involves non-trivially the scalar fields that arise in lower dimensional gauged supergravity.

There is a wide spread believe that the gauged supergravity approach is quite limited to a subset of solutions whose asymptotics is related to near horizon geometries of D-branes. In the particular case of our interest, it is well-known that the D6-brane solution is viewed in eleven dimensions as a Taub-NUT space whereas its near horizon limit is described by an Eguchi-Hanson metric. The former goes asymptotically to $\mathbb{R}^3 \times S^1$ as opposed to the latter that goes as $\mathbb{R}^4$. It is then somehow unexpected to find solutions corresponding to ALC G_2 manifolds in lower dimensional gauged supergravities. Another argument in this line is the following. There is a flop transformation in manifolds of G_2 holonomy with $S^3 \times S^3$ principal orbits that interchanges the fibre S^3 with the base one. This operation, from the point of view of eight-dimensional gauged supergravity, would amount to an exchange between the external sphere and the one where the D6-brane is wrapped. Then, there seems to be no room for the flop within the gauged supergravity approach. So, in particular, flop invariant solutions as the one obtained by Brandhuber, Gomis, Gubser and Gukov [12] would not be obtainable from gauged supergravity. In this paper, we are going to show that this is not the case. It is possible to further extend the twisting conditions in that framework in such a way that general cohomogeneity one metrics of G_2 holonomy with principal orbits $S^3 \times S^3$ turn out to be obtainable within the framework of eight dimensional gauged supergravity.

The generalized twisting procedure that we propose corresponds to a non-trivial embedding of the special lagrangian three-cycle wrapped by the D6-branes in the lower dimensional supergravity. It is important to remark that we are using the name "D6-brane" in quite a loose sense. Meaningly, many G_2 manifolds do not correspond to D6-branes wrapping special lagrangian three-cycles but to the uplift of resolved conifolds with RR fluxes piercing the blown-up S^2. Starting from the general ansatz, we found a set of constraints that neatly reduce it to a six functions one. This makes connection with previous works in the literature where six functions ansatzë were taken as a starting point. The Hitchin system [17] turns out to be an elegant general solution of the constraints. Finally, the flop transformation becomes nicely realized in eight dimensional gauged supergravity. Then, not surprisingly, flop invariant solutions (as that in [12]) emerge in this formalism.

The plan for the rest of the paper is as follows. In section 2 we perform a detailed study of the case of D6-branes on a special lagrangian round three sphere in a manifold with the topology of the complex deformed conifold. We start by analyzing the possible realizations of supersymmetry for the round ansatz and the corresponding generalized twist. Then we formulate our results in terms of the calibrating closed and co-closed three-form associated to manifolds of G_2 holonomy. The analysis carried out for the round case reveals the key points which must be taken into account in the more general case studied in section 3. In this section, a general ansatz with triaxial squashing is considered. The subsequent analysis shows that one must impose certain algebraic constraints on the functions of the ansatz if we require our solution to be supersymmetric.

In section 4 we demonstrate that our formalism provides a realization of the Hitchin system. Some particular cases are studied in section 5, including the flop invariant and the conifold-unification metrics, which were never obtained by using eight dimensional gauged supergravity so far. In section 6 we summarize our results and draw some conclusions. In appendix A we collect the relevant formulae of eight dimensional gauged supergravity. Finally, for completeness in appendix B a lagrangian approach to the round case is presented.

2. D6-branes on the round 8d metric

The first case we will analyze corresponds to a D6-brane wrapping a three-cycle in such a way that the corresponding eight dimensional metric ds_8^2 contains a round three-sphere. Accordingly, we will adopt the following ansatz for the metric:

$$ds_8^2 = e^{2f} dx_{1,3}^2 + e^{2h} d\Omega_3^2 + dr^2 \,, \tag{2.1}$$

where $dx_{1,3}^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$, f and h are functions of the radial coordinate r and $d\Omega_3^2$ is the metric of the unit S^3. It is convenient to parametrize this three-sphere by means of a set of left invariant one-forms w^i, $i = 1, 2, 3$, of the SU(2) group manifold satisfying:

$$dw^i = \frac{1}{2}\epsilon_{ijk} w^j \wedge w^k \,. \tag{2.2}$$

In terms of three Euler angles θ, φ and ψ, the w^i's are:

$$\begin{aligned} w^1 &= \cos\varphi d\theta + \sin\theta \sin\varphi d\psi \,, \\ w^2 &= \sin\varphi d\theta - \sin\theta \cos\varphi d\psi \,, \\ w^3 &= d\varphi + \cos\theta d\psi \,, \end{aligned} \tag{2.3}$$

and $d\Omega_3^2$ is:

$$d\Omega_3^2 = \frac{1}{4}\sum_{i=1}^{3} (w^i)^2 \,. \tag{2.4}$$

In this section we will study some supersymmetric configurations of eight dimensional gauged supergravity [34] whose spacetime metric is of the form displayed in eq. (2.1). The aspects of this theory which are relevant for our analysis have been collected in appendix A. In the configurations studied in the present section, apart from the metric, we will only need to excite the dilatonic scalar ϕ and the SU(2) gauge potential A_μ^i. Actually, we will require that, when uplifted to eleven dimensions, the unwrapped part of the metric be that corresponding to flat four dimensional Minkowski spacetime. This condition determines the following relation between the function f and the field ϕ:

$$f = \frac{\phi}{3} \,. \tag{2.5}$$

(See the uplifting formulae in appendix A).

We will assume that the non-abelian gauge potential A^i_μ has only non-vanishing components along the directions of the S^3. Actually, we will adopt an ansatz in which this field, written as a one-form, is given by:

$$A^i = \left(g - \frac{1}{2}\right) w^i \,, \tag{2.6}$$

with g being a function of the radial coordinate r. Notice that in ref. [25] the value $g = 0$ has been taken. The field strength corresponding to the potential (2.6) is:

$$F^i = g' dr \wedge w^i + \frac{1}{8}(4g^2 - 1)\epsilon^{ijk} w^j \wedge w^k \,. \tag{2.7}$$

By plugging our ansatz of eqs. (2.1), (2.5) and (2.6) in the lagrangian of eight dimensional gauged supergravity, one arrives at an effective problem in which one can find a superpotential and the corresponding first-order domain wall equations. This approach has been followed in appendix B. In this section we will find this same first-order equations by analyzing the supersymmetry transformations of the fermionic fields. As we will verify soon, this last approach will give us the hints we need to extend our analysis to metrics more general than the one written in eq. (2.1).

2.1 Susy analysis

A bosonic configuration of fields is supersymmetric iff the supersymmetry variation of the fermionic fields, evaluated on the configuration, vanishes. In our case the fermionic fields are two pseudo Majorana spinors ψ_λ and χ_i and their supersymmetry transformations are given in appendix A (see eq. (A.6)). In the configurations considered in this section we are not exciting any coset scalar and, therefore, we must take $P_{\mu ij} = 0$ and $T_{ij} = \delta_{ij}$. Moreover, through this paper we shall use the following representation of the Dirac matrices:

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{I} \,, \qquad \hat{\Gamma}^i = \gamma_9 \otimes \sigma^i \,, \tag{2.8}$$

where γ^μ are eight dimensional Dirac matrices, σ^i are Pauli matrices and $\gamma_9 = i\gamma^0\gamma^1 \cdots \gamma^7$ ($\gamma_9^2 = 1$). Actually, in what follows we shall denote by $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ the Dirac matrices along the sphere S^3, by $\{\hat{\Gamma}_1, \hat{\Gamma}_2, \hat{\Gamma}_3\}$ the corresponding matrices along the SU(2) group manifold, whereas $\Gamma_7 \equiv \Gamma_r$ will correspond to the Γ-matrix along the radial direction.

The first-order BPS equations we are trying to find are obtained by requiring that $\delta\chi_i = \delta\psi_\lambda = 0$ for some Killing spinor ϵ, which must satisfy some projection conditions. First of all, we shall impose that:

$$\Gamma_{12}\epsilon = -\hat{\Gamma}_{12}\epsilon \,, \qquad \Gamma_{23}\epsilon = -\hat{\Gamma}_{23}\epsilon \,, \qquad \Gamma_{13}\epsilon = -\hat{\Gamma}_{13}\epsilon \,. \tag{2.9}$$

Notice that in eq. (2.9) the projections along the sphere S^3 and the SU(2) group manifold are related. Actually, only two of these equations are independent and, for example, the last one can be obtained from the first two. Moreover, it follows from (2.9) that:

$$\Gamma_1\hat{\Gamma}_1\epsilon = \Gamma_2\hat{\Gamma}_2\epsilon = \Gamma_3\hat{\Gamma}_3\epsilon. \tag{2.10}$$

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These projections are imposed by the ambient Calabi-Yau three-fold in which the three-cycle lives, from the conditions $J_{ab}\,\epsilon = \Gamma_{ab}\,\epsilon$, where J is the Kähler form. By using eqs. (2.9) and (2.10) to evaluate the right-hand side of (A.6), together with the ansatz for the metric, dilaton and gauge field, one gets some equations which give the radial derivative of ϕ, h and ϵ. Actually, one arrives at the following equation for the radial derivative of the dilaton:

$$\phi'\epsilon = \frac{3}{8}\left[4(1-4g^2)e^{\phi-2h} - e^{-\phi}\right]\Gamma_r\hat{\Gamma}_{123}\epsilon + 3e^{\phi-h}g'\Gamma_1\hat{\Gamma}_1\epsilon\,, \tag{2.11}$$

while the derivative of the function h is:

$$h'\epsilon = 2ge^{-h}\Gamma_1\hat{\Gamma}_1\Gamma_r\hat{\Gamma}_{123}\epsilon - \frac{1}{8}\left[12(1-4g^2)e^{\phi-2h} + e^{-\phi}\right]\Gamma_r\hat{\Gamma}_{123}\epsilon - e^{\phi-h}g'\Gamma_1\hat{\Gamma}_1\epsilon\,. \tag{2.12}$$

Moreover, the radial dependence of the spinor ϵ is determined by:

$$\partial_r\epsilon - \frac{1}{16}\left[4(1-4g^2)e^{\phi-2h} - e^{-\phi}\right]\Gamma_r\hat{\Gamma}_{123}\epsilon + \frac{5}{2}e^{\phi-h}g'\Gamma_1\hat{\Gamma}_1\epsilon = 0\,. \tag{2.13}$$

In order to proceed further, we need to impose some additional condition to the spinor ϵ. It is clear from the right-hand side of eqs. (2.11)-(2.13) that we must specify the action on ϵ of the radial projector $\Gamma_r\hat{\Gamma}_{123}$. The choice made in ref. [25] was to take $g = 0$ and impose the condition $\Gamma_r\hat{\Gamma}_{123}\epsilon = -\epsilon$. It is immediate to verify that in this case our eqs. (2.11)–(2.13) reduce to those obtained in ref. [25]. Here we will not take any *a priori* particular value of $\Gamma_r\hat{\Gamma}_{123}\epsilon$. Instead we will try to determine it in general from eqs. (2.11)–(2.13). Notice that in our approach g will not be constant and, therefore, we will have to find a differential equation which determines it. It is clear from eq. (2.11) that our spinor ϵ must satisfy a relation of the sort:

$$\Gamma_r\hat{\Gamma}_{123}\epsilon = -(\beta + \tilde{\beta}\Gamma_1\hat{\Gamma}_1)\epsilon\,, \tag{2.14}$$

where β and $\tilde{\beta}$ are functions of the radial coordinate r, that can be easily extracted from eq. (2.11), namely:

$$\begin{aligned}\beta &= -\frac{8}{3}\frac{\phi'}{4(1-4g^2)e^{\phi-2h} - e^{-\phi}}\,,\\ \tilde{\beta} &= 8\frac{e^{\phi-h}g'}{4(1-4g^2)e^{\phi-2h} - e^{-\phi}}\,.\end{aligned} \tag{2.15}$$

Equation (2.14) is the kind of radial projection we are looking for. We can get a consistency condition for this projection by noticing that $(\Gamma_r\hat{\Gamma}_{123})^2\epsilon = \epsilon$. Using the fact that $\{\Gamma_r\hat{\Gamma}_{123}, \Gamma_1\hat{\Gamma}_1\} = 0$, this condition is simply:

$$\beta^2 + \tilde{\beta}^2 = 1\,. \tag{2.16}$$

By using in eq. (2.16) the explicit values of β and $\tilde{\beta}$ given in eq. (2.15), one gets:

$$\frac{(\phi')^2}{9} + e^{2\phi-2h}(g')^2 = \frac{1}{64}\left[4(1-4g^2)e^{\phi-2h} - e^{-\phi}\right]^2\,, \tag{2.17}$$

which relates the radial derivatives of ϕ and g. Let us now consider the equation for h' written in eq. (2.12). By using the value of $\Gamma_r\hat{\Gamma}_{123}\epsilon$ given in eq. (2.14), and separating the terms with and without $\Gamma_1\hat{\Gamma}_1\epsilon$, we get two equations:

$$\begin{aligned}
&h' = 2ge^{-h}\tilde{\beta} + \frac{1}{8}\left[12(1-4g^2)e^{\phi-2h} + e^{-\phi}\right]\beta\,,\\
&2ge^{-h}\beta - \frac{1}{8}\left[12(1-4g^2)e^{\phi-2h} + e^{-\phi}\right]\tilde{\beta} + e^{\phi-h}g' = 0\,.
\end{aligned} \tag{2.18}$$

Moreover, by using in the latter the values of β and $\tilde{\beta}$ given in eq. (2.15), we get the following relation between g' and ϕ':

$$g' = -\frac{8g}{3}\frac{e^{2h}\phi'}{4(1-4g^2)e^{2\phi}+e^{2h}}\,. \tag{2.19}$$

Plugging back this equation in the consistency condition (2.17), we can determine ϕ', g', β and $\tilde{\beta}$ in terms of ϕ, g and h. Moreover, by substituting these results on the first equation in (2.18), we get a first-order equation for h. In order to write these equations, let us define the function:

$$K \equiv \sqrt{\left(4(1-2g)^2e^{2\phi}+e^{2h}\right)\left(4(1+2g)^2e^{2\phi}+e^{2h}\right)}\,. \tag{2.20}$$

Then, the BPS equations are:

$$\begin{aligned}
\phi' &= \frac{3}{8}\frac{e^{-2h-\phi}}{K}\left[e^{4h} - 16(1-4g^2)^2e^{4\phi}\right],\\
h' &= \frac{e^{-2h-\phi}}{8K}\left[e^{4h} + 16(1+4g^2)e^{2h+2\phi} + 48(1-4g^2)^2e^{4\phi}\right],\\
g' &= \frac{ge^{-\phi}}{K}\left[4(1-4g^2)e^{2\phi} - e^{2h}\right].
\end{aligned} \tag{2.21}$$

Notice that $g' = g = 0$ certainly solves the last of these equations and, in this case, the first two equations in (2.21) reduce to the ones written in ref. [25]. Moreover, the system (2.21) is identical to that found in ref. [13] by means of the superpotential method (see appendix B). The solutions of (2.21) have been obtained in ref. [13], and they depend on two parameters (see below).

In order to have a clear interpretation of the radial projection we are using, let us notice that, due to the constraint (2.16), we can represent β and $\tilde{\beta}$ as:

$$\beta = \cos\alpha\,, \qquad \tilde{\beta} = \sin\alpha\,. \tag{2.22}$$

Moreover, by substituting the value of ϕ' and g' given by the first-order equations (2.21) into the definition of β and $\tilde{\beta}$ (eq. (2.15)), one arrives at:

$$\tan\alpha = 8g\frac{e^{\phi+h}}{4(1-4g^2)e^{2\phi}+e^{2h}}\,. \tag{2.23}$$

Then, by using the representation (2.22), it is immediate to rewrite eq. (2.14) as:

$$\Gamma_r\hat{\Gamma}_{123}\epsilon = -e^{\alpha\Gamma_1\hat{\Gamma}_1}\epsilon\,. \tag{2.24}$$

Since $\{\Gamma_r\hat{\Gamma}_{123}, \Gamma_1\hat{\Gamma}_1\} = 0$, eq. (2.24) can be solved as:

$$\epsilon = e^{-\frac{1}{2}\alpha\Gamma_1\hat{\Gamma}_1}\epsilon_0\,, \tag{2.25}$$

where ϵ_0 is a spinor satisfying the standard radial projection condition with $\alpha = 0$, i.e.:

$$\Gamma_r\hat{\Gamma}_{123}\epsilon_0 = -\epsilon_0\,. \tag{2.26}$$

To determine completely ϵ_0 we must use eq. (2.13), which dictates the radial dependence of the Killing spinor. Actually, by using the first-order equations (2.21) one can compute $\partial_r\alpha$ from eq. (2.23). The result is remarkably simple, namely:

$$\partial_r\alpha = 6e^{\phi-h}g'\,. \tag{2.27}$$

By using eqs. (2.24) and (2.27) in eq. (2.13), one can verify that ϵ_0 satisfies the equation:

$$\partial_r\epsilon_0 = \frac{\phi'}{6}\epsilon_0\,, \tag{2.28}$$

which can be immediately integrated as:

$$\epsilon_0 = e^{\frac{\phi}{6}}\eta\,, \tag{2.29}$$

with η being a constant spinor. Thus, after collecting all results, it follows that ϵ can be written as:

$$\epsilon = e^{\frac{\phi}{6}}e^{-\frac{1}{2}\alpha\Gamma_1\hat{\Gamma}_1}\eta\,. \tag{2.30}$$

The projections conditions satisfied by η are simply:

$$\Gamma_{12}\hat{\Gamma}_{12}\eta = \eta\,, \qquad \Gamma_{23}\hat{\Gamma}_{23}\eta = \eta\,, \qquad \Gamma_r\hat{\Gamma}_{123}\eta = -\eta\,. \tag{2.31}$$

In order to find out the meaning of the phase α, let us notice that, by using the representation (2.8) for the Γ-matrices, one easily proves that:

$$\Gamma_{x^0\cdots x^3}\Gamma_{123}\Gamma_r\hat{\Gamma}_{123} = -1\,. \tag{2.32}$$

From eqs. (2.9) and (2.32), it is straightforward to verify that the radial projection (2.24) can be written as:

$$\Gamma_{x^0\cdots x^3}\Big(\cos\alpha\Gamma_{123} - \sin\alpha\hat{\Gamma}_{123}\Big)\epsilon = \epsilon\,, \tag{2.33}$$

which is the projection corresponding to a D6-brane wrapped on a three-cycle, which is non-trivially embedded in the two three-spheres, with α measuring the contribution of each sphere. This equation must be understood as seen from the uplifted perspective. The case $\alpha = 0$ corresponds to the D6-brane wrapping a three-sphere that is fully contained in the eight-dimensional spacetime where supergravity lives, and has been studied earlier [25]. Notice that $\alpha = \pi/2$ is not a solution of the system. This is an important consistency check as this would mean that the D6-brane is not wrapping a three-cycle contained in the eight-dimensional spacetime and the twisting would make no sense. However, solutions that asymptotically approach $\alpha = \pi/2$ are possible in principle. In the next subsection we will describe a quantity for which the rotation by the angle α plays an important role.

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2.2 The calibrating three-form

Given a solution of the BPS equations (2.21), one can get an eleven dimensional metric ds^2_{11} by means of the uplifting formula (A.4). The condition (2.5) ensures that the corresponding eleven dimensional manifold is a direct product of four dimensional Minkowski space and a seven dimensional manifold, i.e.:

$$ds^2_{11} = dx^2_{1,3} + ds^2_7 = dx^2_{1,3} + \sum_{A=1}^{7} (e^A)^2 \,, \tag{2.34}$$

where we have written ds^2_7 in terms of a basis of one-forms e^A $(A = 1, \ldots, 7)$. It follows from (A.4) that this basis can be taken as:

$$\begin{aligned}
e^i &= \frac{1}{2} e^{h-\frac{\phi}{3}} w^i \,, && (i = 1,2,3) \,, \\
e^{3+i} &= 2e^{\frac{2\phi}{3}} \left(\tilde{w}^i + \left(g - \frac{1}{2} \right) w^i \right), && (i = 1,2,3) \,, \\
e^7 &= e^{-\frac{\phi}{3}} dr \,.
\end{aligned} \tag{2.35}$$

It is a well-known fact that a manifold of G_2 holonomy is endowed with a calibrating three-form Φ, which must be closed and co-closed with respect to the seven dimensional metric ds^2_7. We shall denote by ϕ_{ABC} the components of Φ in the basis (2.35), namely:

$$\Phi = \frac{1}{3!} \phi_{ABC} e^A \wedge e^B \wedge e^C \,. \tag{2.36}$$

The relation between Φ and the Killing spinors of the metric is also well-known. Indeed, let $\tilde{\epsilon}$ be the Killing spinor uplifted to eleven dimensions, which in terms of ϵ is simply $\tilde{\epsilon} = e^{-\frac{\phi}{6}} \epsilon$. Then, one has:

$$\phi_{ABC} = i \tilde{\epsilon}^\dagger \Gamma_{ABC} \tilde{\epsilon} \,. \tag{2.37}$$

By using the relation between ϵ and the constant spinor η, one can rewrite eq. (2.37) as:

$$\phi_{ABC} = i \eta^\dagger e^{\frac{1}{2}\alpha \Gamma_1 \hat{\Gamma}_1} \Gamma_{ABC} e^{-\frac{1}{2}\alpha \Gamma_1 \hat{\Gamma}_1} \eta \,. \tag{2.38}$$

Let us now denote by $\phi^{(0)}_{ABC}$ the above matrix element when $\alpha = 0$, i.e.:

$$\phi^{(0)}_{ABC} = i \eta^\dagger \Gamma_{ABC} \eta \,. \tag{2.39}$$

It is not difficult to obtain the non-zero matrix elements (2.39). Recall that η is characterized as an eigenvector of the set of projection operators written in eq. (2.31). Thus, if $\mathcal{O}$ is an operator which anticommutes with these projectors, $\mathcal{O}\eta$ and η are eigenvectors of the projectors with different eigenvalues and, therefore, they are orthogonal (i.e. $\eta^\dagger \mathcal{O} \eta = 0$). Moreover, by using the projection conditions (2.31), one can relate the non-vanishing matrix elements to $\eta^\dagger \Gamma_{123} \eta$. If we normalize η such that $i\eta^\dagger \Gamma_{123} \eta = 1$ and if $\hat{i} = i + 3$ for $i = 1,2,3$, one can easily prove that the non-zero $\phi^{(0)}_{ijk}$'s are:

$$\phi^{(0)}_{ijk} = \epsilon_{ijk} \,, \qquad \phi^{(0)}_{i\hat{j}\hat{k}} = -\epsilon_{ijk} \,, \qquad \phi^{(0)}_{7i\hat{j}} = \delta_{ij} \,, \tag{2.40}$$

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as in [35]. By expanding the exponential in (2.38) and using (2.40), it is straightforward to find the different components of Φ for arbitrary α. Actually, one can write the result as:

$$\begin{aligned}\Phi = & \, e^7 \wedge \Big(e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6\Big) + \\ & + \Big(e^1 \cos\alpha + e^4 \sin\alpha\Big) \wedge \Big(e^2 \wedge e^3 - e^5 \wedge e^6\Big) + \\ & + \Big(-e^1 \sin\alpha + e^4 \cos\alpha\Big) \wedge \Big(e^3 \wedge e^5 - e^2 \wedge e^6\Big)\,, \end{aligned} \tag{2.41}$$

which shows that the effect on Φ of introducing the phase α is just a (radial dependent) rotation in the (e^1, e^4) plane (alternatively, the same expression can be written as a rotation in the (e^2, e^5) or (e^3, e^6) plane). As mentioned above, Φ should be closed and co-closed:

$$d\Phi = 0\,, \qquad d *_7 \Phi = 0\,, \tag{2.42}$$

where $*_7$ denotes the Hodge dual in the seven dimensional metric. There is an immediate consequence of this fact which we shall now exploit. Let us denote by p and q the components of Φ along the volume forms of the two three spheres, i.e.:

$$\Phi = p w^1 \wedge w^2 \wedge w^3 + q \tilde{w}^1 \wedge \tilde{w}^2 \wedge \tilde{w}^3 + \cdots\,. \tag{2.43}$$

From the condition $d\Phi = 0$, it follows immediately that p and q must be constants of motion. By plugging the explicit expression of the forms e^A, given in eq. (2.35), on the right-hand side of eq. (2.41), one easily gets p and q in terms of ϕ, h and g. The result is:

$$\begin{aligned} p &= \frac{1}{8}\Big[e^{3h-\phi} - 12 e^{h+\phi}(1-2g)^2\Big]\cos\alpha - \frac{1}{4}(1-2g)\Big[3e^{2h} - 4e^{2\phi}(1-2g)^2\Big]\sin\alpha\,, \\ q &= -8 e^{2\phi}\sin\alpha\,. \end{aligned} \tag{2.44}$$

Notice that $\alpha = 0$ implies $q = 0$ which is precisely the case studied in [25]. By explicit calculation one can check that p and q are constants as a consequence of the BPS equations. Actually, by using (2.21) one can show that, indeed, Φ is closed and co-closed as it should.

To finish this section, let us write down the general solution of the first-order system (2.21), which was obtained in ref. [13]. This solution is expressed in terms of a new radial variable ρ and of the two following functions $Y(\rho)$ and $F(\rho)$:

$$\begin{aligned} Y(\rho) &\equiv \rho^2 - 2(2p+q)\rho + 4p(p+q)\,, \\ F(\rho) &\equiv 3\rho^4 - 8(2p+q)\rho^3 + 24p(p+q)\rho^2 - 16p^2(p+q)^2\,. \end{aligned} \tag{2.45}$$

Notice that $Y(\rho)$ and $F(\rho)$ also depend on the two constants p and q. The seven dimensional metric takes the form:

$$ds_7^2 = F^{-\frac{1}{3}} d\rho^2 + \frac{1}{4} F^{\frac{2}{3}} Y^{-1} (w^i)^2 + F^{-\frac{1}{3}} Y \left(\tilde{w}^i - \left(\frac{1}{2} + \frac{q\rho}{Y}\right) w^i\right)^2\,. \tag{2.46}$$

The analysis of the metrics (2.46) has been carried out in ref. [13]. It turns out that only in three cases ($p = 0$, $q = 0$ and $p = -q$) the metric (2.46) is non-singular. The first two cases are related by the so-called flop transformation, which is a $\mathbb{Z}_2$ action that exchanges w^i and $\tilde{w}^i$, while the $p = -q$ case is flop invariant. It is interesting to point out that, as $g \to 0$ when $\rho \to \infty$, the gauge field (2.6) asymptotically approaches that used in [25] to perform the twisting. This is in line with the fact that the twisting just fixes the value of the gauge field at infinity.

3. D6-branes on a squashed 8d metric

In this section we are going to generalize the analysis performed in section 2 to a much more general situation, in which the eight dimensional metric takes the form:

$$ds_8^2 = e^{\frac{2\phi}{3}} dx_{1,3}^2 + \frac{1}{4} e^{2h_i} (w^i)^2 + dr^2 \,. \tag{3.1}$$

Notice that in the ansatz (3.1) we have already implemented the condition (2.5), which ensures that we are going to have a direct product of four dimensional Minkowski space and a seven dimensional manifold in the uplift to eleven dimensions. As in the previous case, we are going to switch on a SU(2) gauge field potential with components along the squashed S^3. The ansatz we shall adopt for this potential is:

$$A^i = G_i w^i \,, \tag{3.2}$$

which depends on three functions G_1, G_2 and G_3. It should be understood that there is no sum on the right-hand side of eq. (3.2). Moreover, we shall excite coset scalars in the diagonal and, therefore, the corresponding L_α^i matrix will be taken as:

$$L_\alpha^i = \mathrm{diag}(e^{\lambda_1}\,, e^{\lambda_2}, e^{\lambda_3})\,, \qquad \lambda_1 + \lambda_2 + \lambda_3 = 0\,. \tag{3.3}$$

The matrices $P_{\mu ij}$ and $Q_{\mu ij}$ defined in appendix A (eq. (A.1)) are easily evaluated from eqs. (3.2) and (3.3). Written as differential forms, they are:

$$P_{ij} + Q_{ij} = \begin{pmatrix} d\lambda_1 & -A^3 e^{\lambda_{21}} & A^2 e^{\lambda_{31}} \\ A^3 e^{\lambda_{12}} & d\lambda_2 & -A^1 e^{\lambda_{32}} \\ -A^2 e^{\lambda_{13}} & A^1 e^{\lambda_{23}} & d\lambda_3 \end{pmatrix} \,, \tag{3.4}$$

where $\lambda_{ij} \equiv \lambda_i - \lambda_j$ and P_{ij} (Q_{ij}) is the symmetric (antisymmetric) part of the matrix appearing on the right-hand side of (3.4). Notice that our present ansatz depends on nine functions, since there are only two independent λ_i's (see eq. (3.3)). On the other hand, it would be convenient in what follows to define the following combinations of these functions:

$$\begin{aligned} M_1 &\equiv e^{\phi+\lambda_1-h_2-h_3}(G_1 + G_2 G_3)\,, \\ M_2 &\equiv e^{\phi+\lambda_2-h_1-h_3}(G_2 + G_1 G_3)\,, \\ M_3 &\equiv e^{\phi+\lambda_3-h_1-h_2}(G_3 + G_1 G_2)\,. \end{aligned} \tag{3.5}$$

3.1 Susy analysis

With the setup just described, and the experience acquired in the previous section, we will now attack the problem of finding supersymmetric configurations for this more general ansatz. As before, we must guarantee that $\delta\chi_i = \delta\psi_\lambda = 0$ for some spinor ϵ. We begin by imposing again the angular projection condition (2.9). Then, the equation $\delta\chi_1 = 0$ yields:

$$\begin{aligned} \left(\frac{1}{2}\lambda_1' + \frac{1}{3}\phi'\right)\epsilon &= e^{\phi+\lambda_1-h_1} G_1' \Gamma_1 \hat{\Gamma}_1 \epsilon - 2\left[M_1 - \frac{1}{16} e^{-\phi}(e^{2\lambda_1} - e^{2\lambda_2} - e^{2\lambda_3})\right] \Gamma_r \hat{\Gamma}_{123}\epsilon - \\ &- \left[e^{-h_2} G_2 \sinh\lambda_{13} + e^{-h_3} G_3 \sinh\lambda_{12}\right] \Gamma_1 \hat{\Gamma}_1 \Gamma_r \hat{\Gamma}_{123}\epsilon\,, \end{aligned} \tag{3.6}$$

and, obviously, $\delta\chi_2 = \delta\chi_3 = 0$ give rise to other two similar equations which are obtained by permutation of the indices $(1,2,3)$ in eq. (3.6). Adding these three equations and using eq. (2.10) and the fact that $\lambda_1 + \lambda_2 + \lambda_3 = 0$, we get the following equation for ϕ':

$$\begin{aligned}\phi'\epsilon = e^{\phi}\left[e^{\lambda_1 - h_1}G_1' + e^{\lambda_2 - h_2}G_2' + e^{\lambda_3 - h_3}G_3'\right]\Gamma_1\hat{\Gamma}_1\epsilon - \\ - 2\left[M_1 + M_2 + M_3 + \frac{1}{16}e^{-\phi}\Big(e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3}\Big)\right]\Gamma_r\hat{\Gamma}_{123}\epsilon\,. \end{aligned}\tag{3.7}$$

It can be checked that this same equation is obtained from the variation of the gravitino components along the unwrapped directions. Moreover, it follows from eq. (3.7) that $\Gamma_r\hat{\Gamma}_{123}\epsilon$ has the same structure as in eq. (2.14), where now β and $\tilde{\beta}$ are given by:

$$\begin{aligned}\beta &= \frac{8\phi'}{16(M_1 + M_2 + M_3) + e^{-\phi}(e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3})}\,, \\ \tilde{\beta} &= -\frac{8e^{\phi}(e^{\lambda_1 - h_1}G_1' + e^{\lambda_2 - h_2}G_2' + e^{\lambda_3 - h_3}G_3')}{16(M_1 + M_2 + M_3) + e^{-\phi}(e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3})}\,.\end{aligned}\tag{3.8}$$

It is also immediate to see that in the present case β and $\tilde{\beta}$ must also satisfy the constraint (2.16). Thus, in this case we are going to have the same type of radial projection as in the round metric of section 2. Actually, we shall obtain a set of first-order differential equations in terms of β and $\tilde{\beta}$ and then we shall find some consistency conditions which, in particular, allow to determine the values of β and $\tilde{\beta}$. From this point of view it is straightforward to write the equation for ϕ'. Indeed, from the definition of β (eq. (3.8)), one has:

$$\phi' = \left[2(M_1 + M_2 + M_3) + \frac{1}{8}e^{-\phi}(e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3})\right]\beta\,. \tag{3.9}$$

In order to obtain the equation for λ_i' and G_i', let us consider again the equation derived from the condition $\delta\chi_i = 0$ (eq. (3.6)). Plugging the projection condition on the right-hand side of eq. (3.6), using the value of ϕ' displayed in eq. (3.9), and considering the terms without $\Gamma_1\hat{\Gamma}_1$, one gets the equation for λ_1', namely:

$$\begin{aligned}\lambda_1' = \frac{4}{3}\left[2M_1 - M_2 - M_3 - \frac{1}{8}e^{-\phi}\Big(2e^{2\lambda_1} - e^{2\lambda_2} - e^{2\lambda_3}\Big)\right]\beta - \\ - 2\left[e^{-h_2}G_2\sinh\lambda_{13} + e^{-h_3}G_3\sinh\lambda_{12}\right]\tilde{\beta}\,.\end{aligned}\tag{3.10}$$

while the terms with $\Gamma_1\hat{\Gamma}_1$ of eq. (3.6) yield the equation for G_1':

$$\begin{aligned}e^{\phi + \lambda_1 - h_1}G_1' = \left[-2M_1 + \frac{1}{8}e^{-\phi}(e^{2\lambda_1} - e^{2\lambda_2} - e^{2\lambda_3})\right]\tilde{\beta} - \\ - \left[e^{-h_2}G_2\sinh\lambda_{13} + e^{-h_3}G_3\sinh\lambda_{12}\right]\beta\,.\end{aligned}\tag{3.11}$$

By cyclic permutation of eqs. (3.10) and (3.11) one obtains the first-order differential equations of λ_i' and G_i' for $i = 2, 3$.

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It remains to obtain the equation for h_i'. With this purpose let us consider the supersymmetric variation of the gravitino components along the sphere. One gets:

$$\begin{aligned}
h_1'\epsilon = & -\frac{1}{3}e^{\phi}\left[5e^{\lambda_1-h_1}G_1' - e^{\lambda_2-h_2}G_2' - e^{\lambda_3-h_3}G_3'\right]\Gamma_1\hat{\Gamma}_1\epsilon - \\
& -\frac{1}{3}\left[2(M_1-5M_2-5M_3)+\frac{1}{8}e^{-\phi}(e^{2\lambda_1}+e^{2\lambda_2}+e^{2\lambda_3})\right]\Gamma_r\hat{\Gamma}_{123}\epsilon - \\
& -\left[\frac{e^{2h_1}-e^{2h_2}-e^{2h_3}}{e^{h_1+h_2+h_3}} - 2e^{-h_1}G_1\cosh\lambda_{23}\right]\Gamma_1\hat{\Gamma}_1\Gamma_r\hat{\Gamma}_{123}\epsilon\,,
\end{aligned} \tag{3.12}$$

and two other equations obtained by cyclic permutation. By considering the terms without $\Gamma_1\hat{\Gamma}_1$ in eq. (3.12) we get the desired first-order equation for h_1', namely:

$$\begin{aligned}
h_1' = & \frac{1}{3}\left[2(M_1-5M_2-5M_3)+\frac{1}{8}e^{-\phi}(e^{2\lambda_1}+e^{2\lambda_2}+e^{2\lambda_3})\right]\beta - \\
& -\left[\frac{e^{2h_1}-e^{2h_2}-e^{2h_3}}{e^{h_1+h_2+h_3}} - 2e^{-h_1}G_1\cosh\lambda_{23}\right]\tilde{\beta}\,.
\end{aligned} \tag{3.13}$$

On the other hand, the terms with $\Gamma_1\hat{\Gamma}_1$ of eq. (3.12) give rise to new equations for the G_i''s:

$$\begin{aligned}
e^{\phi} & \left[5e^{\lambda_1-h_1}G_1' - e^{\lambda_2-h_2}G_2' - e^{\lambda_3-h_3}G_3'\right] = \\
& = \left[2(M_1-5M_2-5M_3)+\frac{1}{8}e^{-\phi}(e^{2\lambda_1}+e^{2\lambda_2}+e^{2\lambda_3})\right]\tilde{\beta} + \\
& \quad + 3\left[\frac{e^{2h_1}-e^{2h_2}-e^{2h_3}}{e^{h_1+h_2+h_3}} - 2e^{-h_1}G_1\cosh\lambda_{23}\right]\beta\,.
\end{aligned} \tag{3.14}$$

This equation (and the other two obtained by cyclic permutation) must be compatible with the equation for G_i' written in eq. (3.11). Actually, by substituting in eq. (3.14) the value of G_i' given by eq. (3.11), and by combining appropriately the equations so obtained, we arrive at three algebraic relations of the type:

$$\mathcal{A}_i\beta - \mathcal{B}_i\tilde{\beta} = 0\,, \tag{3.15}$$

where $\mathcal{A}_1$ and $\mathcal{B}_1$ are given by:

$$\begin{aligned}
\mathcal{A}_1 & = e^{h_1-h_2-h_3} + e^{\lambda_1-\lambda_3-h_2}G_2 + e^{\lambda_1-\lambda_2-h_3}G_3\,, \\
\mathcal{B}_1 & = -4M_1 + \frac{1}{4}e^{-\phi+2\lambda_1}\,,
\end{aligned} \tag{3.16}$$

while the values of $\mathcal{A}_i$ and $\mathcal{B}_i$ for $i=2,3$ are obtained from (3.16) by cyclic permutation. Notice that the above relations do not involve derivatives of the fields and, in particular, they allow to obtain the values of β and $\tilde{\beta}$. Indeed, by using the constraint $\beta^2+\tilde{\beta}^2=1$, and eq. (3.15) for $i=1$, we get:

$$\beta = \frac{\mathcal{B}_1}{\sqrt{\mathcal{A}_1^2+\mathcal{B}_1^2}}\,, \qquad \tilde{\beta} = \frac{\mathcal{A}_1}{\sqrt{\mathcal{A}_1^2+\mathcal{B}_1^2}}\,. \tag{3.17}$$

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Moreover, it is clear from (3.15) that the $\mathcal{A}_i$'s and $\mathcal{B}_i$'s must satisfy the following consistency conditions:

$$\mathcal{A}_i \mathcal{B}_j = \mathcal{A}_j \mathcal{B}_i \,, \qquad (i \neq j)\,. \tag{3.18}$$

Equation (3.18) gives two independent algebraic constraints that the functions of our generic ansatz must satisfy if we demand it to be a supersymmetric solution. Notice that these constraints are trivially satisfied in the round case of section 2. On the other hand, if we adopt the radial projection of refs. [25, 28], i.e. when $\beta = 1$ and $\tilde{\beta} = 0$, they imply that $\mathcal{A}_i = 0$ (see eq. (3.15)), this leading precisely to the values of the SU(2) gauge potential used in those references. Moreover, by using eq. (3.15), the differential equation satisfied by the G_i's can be simplified. One gets:

$$\begin{aligned} e^{\phi+\lambda_1-h_1} G_1 &= \frac{1}{2}\Big[e^{h_1-h_2-h_3} + e^{\lambda_3-\lambda_1-h_2} G_2 + e^{\lambda_2-\lambda_1-h_3} G_3\Big]\beta - \\ &\quad - \frac{e^{-\phi}}{8}\Big(e^{2\lambda_2} + e^{2\lambda_3}\Big)\tilde{\beta}\,, \end{aligned} \tag{3.19}$$

and similar expressions for G_2 and G_3.

Let us now parametrize β and $\tilde{\beta}$ as in eq. (2.22), i.e. $\beta = \cos\alpha$, $\tilde{\beta} = \sin\alpha$. Then, it follows from eq. (3.17) that one has:

$$\tan\alpha = \frac{\mathcal{A}_1}{\mathcal{B}_1} = \frac{\mathcal{A}_2}{\mathcal{B}_2} = \frac{\mathcal{A}_3}{\mathcal{B}_3}\,. \tag{3.20}$$

Notice that by taking $\alpha = 0$, eq. (3.19) precisely leads to the expression for the gauge field in terms of scalar fields used in [28] to perform the twisting. Moreover, the radial projection condition can be written as in eq. (2.24) and, thus, the natural solution to the Killing spinor equations is just the one written in eq. (2.30), where η is a constant spinor satisfying the conditions (2.31). To check that this is the case, one can plug the expression of ϵ given in eq. (2.30) in the equation arising from the supersymmetric variation of the radial component of the gravitino. It turns out that this equation is satisfied provided α satisfies the equation:

$$\partial_r \alpha = -\Big[4\Big(M_1 + M_2 + M_3\Big) + \frac{1}{4} e^{-\phi}\Big(e^{2\lambda_1} + e^{2\lambda_2} + e^{2\lambda_3}\Big)\Big]\sin\alpha\,. \tag{3.21}$$

In general, this equation for α does not follow from the first-order equations and the algebraic constraints we have found. Actually, by using the value of α given in eq. (3.20) and the first-order equations to evaluate the left-hand side of eq. (3.21), we could derive a third algebraic constraint. However, this new constraint is rather complicated. Happily, we will not need to do this explicitly since eq. (3.21) will serve to our purposes.

3.2 The calibrating three-form

In order to find the calibrating three-form Φ in this case, let us take the following vierbein basis:

$$\begin{aligned} e^i &= \frac{1}{2} e^{h_i - \frac{\phi}{3}} w^i\,, && (i = 1,2,3)\,, \\ e^{3+i} &= 2 e^{\frac{2\phi}{3}+\lambda_i}\Big(\tilde{w}^i + G_i w^i\Big)\,, && (i = 1,2,3)\,, \\ e^7 &= e^{-\frac{\phi}{3}} dr\,, \end{aligned} \tag{3.22}$$

which is the natural one for the uplifted metric. The different components of Φ can be computed by using eq. (2.36) and it is obvious from the form of the projection that the result is just the one given in eq. (2.41), where now α is given by eq. (3.20) and the one-forms e^A are the ones written in eq. (3.22). If, as in eq. (2.43), p and q denote the components of Φ along the two three spheres, it follows from the closure of Φ that p and q should be constants of motion. By plugging the expressions of the e^A's, taken from eq. (3.22), on the right-hand side of eq. (2.41), one can find the explicit expressions of p and q. The result is:

$$\begin{aligned} p &= \frac{1}{8}\Big[e^{h_1+h_2+h_3-\phi} - 16e^{\phi}\Big(e^{h_1-\lambda_1}G_2G_3 + e^{h_2-\lambda_2}G_1G_3 + e^{h_3-\lambda_3}G_1G_2\Big)\Big]\cos\alpha + \\ &\quad + \frac{1}{2}\Big[e^{h_2+h_3+\lambda_1}G_1 + e^{h_1+h_3+\lambda_2}G_2 + e^{h_1+h_2+\lambda_3}G_3 - 16e^{2\phi}G_1G_2G_3\Big]\sin\alpha\,, \\ q &= -8e^{2\phi}\sin\alpha\,. \end{aligned} \tag{3.23}$$

It is a simple exercise to verify that, when restricted to the round case studied above, the expressions of p and q given in eq. (3.23) coincide with those written in eq. (2.44). Moreover, the proof of the constancy of p and q can be performed by combining appropriately the first-order equations and the constraints. Actually, by using eq. (3.9) to compute the radial derivative of q in eq. (3.23), it follows that the condition $\partial_r q = 0$ is equivalent to eq. (3.21). Although the proof of $\partial_r p = 0$ is much more involved, one can demonstrate that p is indeed constant by using the BPS equations and the constraints (3.18) and (3.21).

4. The Hitchin system

A simple counting argument can be used to determine the number of independent functions left out by the constraints. Indeed, we have already mentioned that our ansatz depends on nine functions. However, we have found two constraints in eq. (3.18) and one extra condition which fixes $\partial_r\alpha$ in eq. (3.21). It is thus natural to think that the number of independent functions is six and, thus, in principle, one should be able to express the metric and the BPS equations in terms of them. By looking at the complicated form of the first-order equations and constraints one could be tempted to think that this is a hopeless task. However, we will show that this is not the case and that there exists a set of variables, which are precisely those introduced by Hitchin in ref. [17], in which the BPS equations drastically simplify. These equations involve the constants p and q just discussed, together with the components of the calibrating three-form Φ. Actually, following refs. [17, 16, 18], we shall parametrize Φ as:

$$\Phi = e^7 \wedge \omega + \rho\,, \tag{4.1}$$

where the two-form ω is given in terms of three functions y_i as:

$$\omega = \sqrt{\frac{y_2y_3}{y_1}}w^1\wedge\tilde{w}^1 + \sqrt{\frac{y_3y_1}{y_2}}w^2\wedge\tilde{w}^2 + \sqrt{\frac{y_1y_2}{y_3}}w^3\wedge\tilde{w}^3\,, \tag{4.2}$$

and ρ is a three-form which depends on another set of three functions x_i, namely:

$$\begin{aligned} \rho &= pw^1\wedge w^2\wedge w^3 + q\tilde{w}^1\wedge\tilde{w}^2\wedge\tilde{w}^3 + \\ &\quad + x_1\Big(w^1\wedge\tilde{w}^2\wedge\tilde{w}^3 - w^2\wedge w^3\wedge\tilde{w}^1\Big) + \text{cyclic}\,. \end{aligned} \tag{4.3}$$

Notice that the terms appearing in ω are precisely those which follow from our expression (2.41) for Φ. Moreover, by plugging on the right-hand side of eq. (2.41) the relation (3.22) between the one-forms e^A and the SU(2) left invariant forms, one can find the explicit relation between the new and old variables, namely:

$$\begin{aligned} y_1 &= e^{\frac{2\phi}{3}+h_2+h_3-\lambda_1}\,, \\ x_1 &= -2\Big[e^{\phi+h_1-\lambda_1}\cos\alpha + 4e^{2\phi}G_1\sin\alpha\Big]\,, \end{aligned} \tag{4.4}$$

and cyclically in $(1,2,3)$. Notice that the coefficients of $w^1\wedge\tilde{w}^2\wedge\tilde{w}^3$ and of $-w^2\wedge w^3\wedge\tilde{w}^1$ in the expression (4.3) of ρ must be necessarily equal if Φ is closed. Actually, by computing the latter in our formalism, we get an alternative expression for the x_i's. This other expression is:

$$\begin{aligned} x_1 &= 2\Big[e^{h_3-\lambda_3}G_2 + e^{h_2-\lambda_2}G_3\Big]e^{\phi}\cos\alpha + \\ &\quad + \left[8e^{2\phi}G_2G_3 - \frac{1}{2}e^{\lambda_1+h_2+h_3}\right]\sin\alpha\,, \end{aligned} \tag{4.5}$$

and cyclically in $(1,2,3)$. As a matter of fact, these two alternative expressions for the x_i's are equal as a consequence of the constraints (3.15). In fact, we can regard eqs. (3.15) and (3.21) as conditions needed to ensure the closure of Φ. On the other hand, by using, at our convenience, eqs. (4.4) and (4.5), one can prove the following useful relations:

$$\begin{aligned} \frac{x_2x_3-px_1}{y_1} &= \frac{1}{4}e^{2h_1-\frac{2\phi}{3}} + 4e^{\frac{4\phi}{3}+2\lambda_1}G_1^2\,, \\ \frac{x_1^2-x_2^2-x_3^2-pq}{y_1} &= 8e^{\frac{4\phi}{3}+2\lambda_1}G_1\,, \\ \frac{x_2x_3+qx_1}{y_1} &= 4e^{\frac{4\phi}{3}+2\lambda_1}\,, \end{aligned} \tag{4.6}$$

and cyclically in $(1,2,3)$. As a first application of eq. (4.6), let us point out that, making use of this equation, one can easily invert the relation (4.4). The result is:

$$\begin{aligned} e^{2\phi} &= \frac{1}{8}\frac{(qx_1+x_2x_3)^{1/2}(qx_2+x_1x_3)^{1/2}(qx_3+x_1x_2)^{1/2}}{\sqrt{y_1y_2y_3}}\,, \\ e^{2\lambda_1} &= \frac{(y_2y_3)^{1/3}}{(y_1)^{2/3}}\frac{(qx_1+x_2x_3)^{2/3}}{(qx_2+x_1x_3)^{1/3}(qx_3+x_1x_2)^{1/3}}\,, \\ e^{2h_1} &= 2\frac{(y_2y_3)^{5/6}}{(y_1)^{1/6}}\frac{(qx_2+x_1x_3)^{1/6}(qx_3+x_1x_2)^{1/6}}{(qx_1+x_2x_3)^{5/6}}\,, \\ G_1 &= \frac{1}{2}\frac{x_1^2-x_2^2-x_3^2-pq}{qx_1+x_2x_3}\,, \end{aligned} \tag{4.7}$$

and cyclically in $(1,2,3)$. Moreover, in order to make contact with the formalism of refs. [17, 18], let us define now the following "potential":

$$\begin{aligned} U &\equiv p^2q^2 + 2pq(x_1^2+x_2^2+x_3^2) + 4(p-q)x_1x_2x_3 + \\ &\quad + x_1^4+x_2^4+x_3^4 - 2x_1^2x_2^2 - 2x_2^2x_3^2 - 2x_3^2x_1^2\,. \end{aligned} \tag{4.8}$$

A straightforward calculation shows that U can be rewritten as:

$$U = \frac{1}{3}(x_1^2 - x_2^2 - x_3^2 - pq)^2 - \frac{4}{3}(x_2x_3 + qx_1)(x_2x_3 - px_1) + \\ + \text{cyclic permutations}\,. \tag{4.9}$$

By using (4.6) to evaluate the right-hand side of eq. (4.9), together with the definition of the y_i's written in eq. (4.4), one easily verifies that U is given by:

$$U = -4y_1y_2y_3\,. \tag{4.10}$$

It is important to stress the fact that in the general Hitchin formalism the relation (4.10) is a constraint, whereas here this equation is just an identity which follows from the definitions of p, q, x_i and y_i. Another important consequence of the identities (4.6) is the form of the metric in the new variables. Indeed, it is immediate from eqs. (3.22) and (4.6) to see that the seven dimensional metric ds_7^2 takes the form:

$$\begin{aligned} ds_7^2 = {}& dt^2 + \\ & + \frac{1}{y_1}\Big[\Big(x_2x_3 - px_1\Big)(w^1)^2 + \Big(x_1^2 - x_2^2 - x_3^2 - pq\Big)w^1\tilde{w}^1 + \Big(x_2x_3 + qx_1\Big)(\tilde{w}^1)^2\Big] + \\ & + \frac{1}{y_2}\Big[\Big(x_3x_1 - px_2\Big)(w^2)^2 + \Big(x_2^2 - x_3^2 - x_1^2 - pq\Big)w^2\tilde{w}^2 + \Big(x_3x_1 + qx_2\Big)(\tilde{w}^2)^2\Big] + \\ & + \frac{1}{y_3}\Big[\Big(x_1x_2 - px_3\Big)(w^3)^2 + \Big(x_3^2 - x_1^2 - x_2^2 - pq\Big)w^3\tilde{w}^3 + \Big(x_1x_2 + qx_3\Big)(\tilde{w}^3)^2\Big]\,, \end{aligned} \tag{4.11}$$

where $dt^2 = e^{-2\phi/3}dr^2$.

It remains to determine the first-order system of differential equations satisfied by the new variables. First of all, recall that, in the old variables, the BPS equations depend on the phase α. Actually, from the expression of q (eq. (3.23)), and the first equation in (4.7), one can easily determine $\sin\alpha$, whereas $\cos\alpha$ can be obtained from the second equation in (4.4). The result is:

$$\begin{aligned} \sin\alpha &= -q\frac{\sqrt{y_1y_2y_2}}{(qx_1 + x_2x_3)^{1/2}(qx_2 + x_1x_3)^{1/2}(qx_3 + x_1x_2)^{1/2}}\,, \\ \cos\alpha &= -\frac{2x_1x_2x_3 + q(x_1^2 + x_2^2 + x_3^2) + pq^2}{2(qx_1 + x_2x_3)^{1/2}(qx_2 + x_1x_3)^{1/2}(qx_3 + x_1x_2)^{1/2}}\,. \end{aligned} \tag{4.12}$$

As a check of eq. (4.12) one can easily verify that $\sin^2\alpha + \cos^2\alpha = 1$ as a consequence of the relation (4.10). It is now straightforward to compute the derivatives of x_i and y_i. Indeed, one can differentiate eq. (4.4) and use eqs. (3.9), (3.10), (3.13), (3.19) and (3.21) to evaluate the result in the old variables. This result can be converted back to the new variables by means of eqs. (4.7) and (4.12). The final result of these calculations is remarkably simple, namely:

$$\begin{aligned} \dot{x}_1 &= -\sqrt{\frac{y_2y_3}{y_1}}\,, \\ \dot{y}_1 &= \frac{pqx_1 + (p-q)x_2x_3 + x_1(x_1^2 - x_2^2 - x_3^2)}{\sqrt{y_1y_2y_3}}\,, \end{aligned} \tag{4.13}$$

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and cyclically in $(1,2,3)$. In eq. (4.13) the dot denotes derivative with respect to the variable t defined after eq. (4.11). The first-order system (4.13) is, with our notations, the one derived in refs. [17, 18]. Indeed, one can show that the equations satisfied by the x_i's are a consequence of the condition $d\Phi = 0$, whereas, if the seven dimensional Hodge dual is computed with the metric (4.11), then $d *_7 \Phi = 0$ implies the first-order equations for the y_i's. Therefore, we have shown that eight dimensional gauged supergravity provides an explicit realization of the Hitchin formalism for general values of the constants p and q. Notice that a non-zero phase α is needed in order to get a system with $q \neq 0$. Recall (see eq. (2.33)) that the phase α parametrizes the tilting of the three cycle on which the D6-brane is wrapped with respect to the three sphere of the eight dimensional metric. Notice that the analysis of [28] corresponds to the case $q = \alpha = 0$.

Let us finally point out that the first-order equations (4.13) are invariant if we change the constants (p,q) by $(-q,-p)$. In the metric (4.11) this change is equivalent to the exchange of w^i and $\tilde{w}^i$, i.e. of the two S^3 of the principal orbits of the cohomogeneity one metric (4.11). As mentioned above, this is the so-called flop transformation. Thus, we have proved that:

$$w^i \leftrightarrow \tilde{w}^i \Longleftrightarrow (p,q) \leftrightarrow (-q,-p)\,. \tag{4.14}$$

Notice that the three-form Φ given in eqs. (4.1)-(4.3) changes its sign when both $(w^i, \tilde{w}^i)$ and (p,q) are transformed as in eq. (4.14).

5. Some particular cases

With the kind of ansatz we are adopting for the eight-dimensional solutions, the corresponding eleven dimensional metrics are of the type:

$$ds^2_{11} = dx^2_{1,3} + B_i^2 (w^i)^2 + D_i^2 (\tilde{w}^i + G_i w^i)^2 + dt^2\,, \tag{5.1}$$

where the coefficients B_i, D_i and the variable t are related to eight dimensional quantities as follows:

$$B_i^2 = \frac{1}{4} e^{2h_i - \frac{2\phi}{3}}\,, \qquad D_i^2 = 4 e^{\frac{4\phi}{3} + 2\lambda_i}\,, \qquad dt^2 = e^{-\frac{2\phi}{3}} dr^2\,. \tag{5.2}$$

Moreover, we have found that, for a supersymmetric solution, the nine functions appearing in the metric are not independent but rather they are related by some algebraic constraints which are, in general, quite complicated. Notice that, in this case, the gauged supergravity approach forces the six function ansatz, this possibly clarifying the reasons behind this *a priori* requirement in previous cases in the literature. To illustrate this point, let us write eq. (3.18) in terms of B_i, D_i and G_i. One gets:

$$\begin{aligned}&\Big[B_1 D_2^2 D_1 G_2 - (1 \leftrightarrow 2)\Big] D_3^2 (1 - G_3^2) = \\ &\qquad = B_3 D_2 \left[B_1 B_3 D_1 D_2 G_2 + D_1^2 D_2 D_3 G_1^2 + B_1^2 D_2 D_3\right] - (1 \leftrightarrow 2)\,,\end{aligned} \tag{5.3}$$

and cyclically in $(1,2,3)$. In addition, we must ensure that eq. (3.21) is also satisfied. Despite the terrifying aspect of eq. (5.3), it is not hard to find expression for, say, G_2 and G_3 in terms of the remaining functions. Moreover, we will be able to find some particular solutions, which correspond to the different cohomogeneity one metrics with $S^3 \times S^3$ principal orbits and SU(2)$\times$ SU(2) isometry which have been studied in the literature.

5.1 The $q = 0$ solution

The simplest way of solving the constraints imposed by supersymmetry is by taking $q = 0$. A glance at the second equation in (3.23) reveals that in this case $\sin\alpha = 0$ and, thus, $\beta = 1$, $\tilde{\beta} = 0$. Notice, first of all, that this is a consistent solution of eq. (3.21). Moreover, it follows from eq. (3.15) that one must have:

$$\mathcal{A}_i = 0\,. \tag{5.4}$$

By combining the three conditions (5.4) it is easy to find the values of the gauge field components G_i in terms of the other functions B_i and D_i [28]. One gets:

$$G_1 = \frac{1}{2}\frac{D_2 D_3}{B_2 B_3}\left[\left(\frac{B_1}{D_1}\right)^2 - \left(\frac{B_2}{D_2}\right)^2 - \left(\frac{B_3}{D_3}\right)^2\right], \tag{5.5}$$

and cyclically in $(1,2,3)$, which is precisely the result of [28]. This is the solution of the constraints we were looking for. One can check that, assuming that the G_i's are given by eq. (5.5), then eq. (3.19) for G_i' is satisfied if eqs. (3.9), (3.10) and (3.13) hold. Thus, eq. (5.5) certainly gives a consistent truncation of the first-order differential equations. On the other hand, by using the value of the G_i's given in eq. (5.5), one can eliminate them and obtain a system of first-order equations for the remaining functions B_i and D_i. These equations are:

$$\begin{aligned}
\dot{B}_1 &= -\frac{D_2}{2B_3}(G_2 + G_1 G_3) - \frac{D_3}{2B_2}(G_3 + G_1 G_2)\,,\\
\dot{D}_1 &= \frac{D_1^2}{2B_2 B_3}(G_1 + G_2 G_3) + \frac{1}{2D_2 D_3}(D_2^2 + D_3^2 - D_1^2)\,,
\end{aligned} \tag{5.6}$$

together with the other permutations of the indices $(1,2,3)$. In (5.6) the G_i's are are the functions of B_i and D_i displayed in eq. (5.5). The constant p can be immediately obtained from (3.23), namely:

$$p = B_1 B_2 B_3 - B_1 D_2 D_3 G_2 G_3 - B_2 D_1 D_3 G_1 G_3 - B_3 D_1 D_2 G_1 G_2\,. \tag{5.7}$$

Let us now give the Hitchin variables in this case. By taking $\alpha = 0$ on the right-hand side of (4.4) and using the relation (5.2), one readily arrives at:

$$x_1 = -B_1 D_2 D_3\,, \qquad y_1 = B_2 B_3 D_2 D_3\,. \tag{5.8}$$

The values of the other x_i and y_i are obtained by cyclic permutation. As a verification of these expressions, it is not difficult to demonstrate, by using eq. (5.6), that the functions x_i and y_i of eq. (5.8) satisfy the first-order equations (4.13) for $q = 0$. Finally, let us point out that, by means of a flop transformation, one can pass from the $q = 0$ metric described above to a metric with $p = 0$.

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5.2 The flop invariant solution

It is also possible to solve our constraints by requiring that the metric be invariant under the $\mathbb{Z}_2$ flop transformation $w^i \leftrightarrow \tilde{w}^i$. It follows from eq. (4.14) that, in this case, we must necessarily have $p = -q$. Moreover, it is also clear that the forms w^i and $\tilde{w}^i$ must enter the metric in the combinations $(w^i + \tilde{w}^i)^2$ and $(w^i - \tilde{w}^i)^2$, which are the only quadratic combinations which are invariant under the flop transformation. Thus the metric we are seeking must be of the type:

$$ds^2_{11} = dx^2_{1,3} + a_i^2(w^i - \tilde{w}^i)^2 + b_i^2(w^i + \tilde{w}^i)^2 + dt^2\,, \tag{5.9}$$

where a_i and b_i are functions which obey some system of first-order differential equations to be determined. In general [13], a metric of the type written in eq. (5.1) can be put in the form (5.9) only if G_i, B_i and D_i satisfy the following relation:

$$G_i^2 = 1 - \frac{B_i^2}{D_i^2}\,. \tag{5.10}$$

It is easy to show that our constraints are solved for G_i given as in eq. (5.10). Indeed, after some calculations, one can rewrite the constraint (3.18) for $i = 1$ and $j = 2$ as:

$$\begin{aligned}&\left(1 - \frac{1}{16}e^{-2\phi+2h_1-2\lambda_1} - G_1^2\right)e^{-2\lambda_3} - \left(1 - \frac{1}{16}e^{-2\phi+2h_2-2\lambda_2} - G_2^2\right)e^{-2\lambda_3} + \\ &+ \left(1 - \frac{1}{16}e^{-2\phi+2h_3-2\lambda_3} - G_3^2\right)\left[G_2 e^{h_1-h_3+\lambda_2} - G_1 e^{h_2-h_3+\lambda_1}\right] = 0\,,\end{aligned} \tag{5.11}$$

which is clearly solved for:

$$G_i^2 = 1 - \frac{1}{16}e^{-2\phi+2h_i-2\lambda_i}\,. \tag{5.12}$$

Similarly, one can verify that eq. (5.12) also solves eq. (3.18) for the remaining values of i and j. After taking into account the identifications (5.2), we easily conclude that the solution (5.12) coincides with the condition (5.10) and, thus, it corresponds to $\mathbb{Z}_2$ invariant metric of the type (5.9). Moreover, it can be checked that the relation (5.12) gives a consistent truncation of the first-order differential equations found in section 3 and that eq. (3.21) is also satisfied. On the other hand, the identification of the a_i and b_i functions with the ones corresponding to 8d gauged supergravity is easily established by comparing the uplifted metric with (5.9), namely:

$$\begin{aligned} dr &= e^{\frac{\phi}{3}}dt\,, \\ e^{2h_i - \frac{2\phi}{3}} &= 16\frac{a_i^2 b_i^2}{a_i^2 + b_i^2}\,, \\ e^{\frac{4\phi}{3}+2\lambda_i} &= \frac{1}{4}(a_i^2 + b_i^2)\,. \end{aligned} \tag{5.13}$$

This relation allows to obtain ϕ, λ_i and h_i in terms of a_i and b_i:

$$e^{2\phi} = \frac{1}{8}\prod_i (a_i^2 + b_i^2)^{\frac{1}{2}}\,,$$

$$
\begin{aligned}
e^{2\lambda_i} &= \frac{a_i^2 + b_i^2}{\prod_j (a_j^2 + b_j^2)^{\frac{1}{3}}}\,, \\
e^{2h_i} &= 8\frac{a_i^2 b_i^2}{a_i^2 + b_i^2}\prod_j (a_j^2 + b_j^2)^{\frac{1}{6}}\,,
\end{aligned} \tag{5.14}
$$

while G_i in terms of the a_i and b_i is given by:

$$
G_i = \frac{b_i^2 - a_i^2}{b_i^2 + a_i^2}\,. \tag{5.15}
$$

The inverse relation is also useful:

$$
a_i^2 = 2e^{\frac{4\phi}{3}+2\lambda_i}(1 - G_i)\,, \qquad b_i^2 = 2e^{\frac{4\phi}{3}+2\lambda_i}(1 + G_i)\,, \tag{5.16}
$$

where G_i is the function of ϕ, h_i and λ_i written in eq. (5.12). By using eqs. (5.14) and (5.15) one can obtain the values of $\cos\alpha$ and $\sin\alpha$ for this case. One gets:

$$
\begin{aligned}
\cos\alpha &= \frac{b_1a_2a_3 + a_1b_2a_3 + a_1a_2b_3 - b_1b_2b_3}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)(a_3^2+b_3^2)}}\,, \\
\sin\alpha &= \frac{a_1b_2b_3 + b_1a_2b_3 + b_1b_2a_3 - a_1a_2a_3}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)(a_3^2+b_3^2)}}\,.
\end{aligned} \tag{5.17}
$$

Moreover, by differentiating eq. (5.16) and using the first-order equations of section 3, together with eqs. (5.14) and (5.17), one can find the BPS equations in the a_i and b_i variables. They are:

$$
\begin{aligned}
\dot{a}_1 &= -\frac{a_1^2}{4a_2b_3} - \frac{a_1^2}{4a_3b_2} + \frac{a_2}{4b_3} + \frac{b_2}{4a_3} + \frac{a_3}{4b_2} + \frac{b_3}{4a_2}\,, \\
\dot{b}_1 &= -\frac{b_1^2}{4a_2a_3} + \frac{b_1^2}{4b_2b_3} - \frac{b_2}{4b_3} + \frac{a_2}{4a_3} - \frac{b_3}{4b_2} + \frac{a_3}{4a_2}\,,
\end{aligned} \tag{5.18}
$$

and cyclically for the other a_i's and b_i's. These are precisely the equations found in ref. [12] for this type of metrics. Moreover, it is now straightforward to compute the constants p and q in this case. Indeed, by substituting eqs. (5.14), (5.15) and (5.17) on the right-hand side of eq. (3.23), one easily proves that:

$$
p = -q = a_1b_2b_3 + b_1a_2b_3 + b_1b_2a_3 - a_1a_2a_3\,. \tag{5.19}
$$

Similarly, from eq. (4.4) one can find the Hitchin variables in terms of the a_i's and b_i's. The result for x_1 and y_1 is:

$$
\begin{aligned}
x_1 &= a_1b_2b_3 - b_1a_2b_3 - b_1b_2a_3 - a_1a_2a_3\,, \\
y_1 &= 4a_2a_3b_2b_3\,,
\end{aligned} \tag{5.20}
$$

while the expressions of x_2, x_3, y_2 and y_3 are obtained from (5.20) by cyclic permutations.

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5.3 The conifold-unification metrics

There exist a class of G_2 metrics with $S^3 \times S^3$ principal orbits which have an extra U(1) isometry and generic values of p and q. They are the so-called conifold-unification metrics and they were introduced in ref. [14] as a unification, via M-theory, of the deformed and resolved conifolds. Following ref. [14], let us parametrize them as:

$$
\begin{aligned}
ds_7^2 = {} & a^2\Big[\Big(\tilde{w}^1+\mathcal{G}w^1\Big)^2+\Big(\tilde{w}^2+\mathcal{G}w^2\Big)^2\Big]+b^2\Big[\Big(\tilde{w}^1-\mathcal{G}w^1\Big)^2+\Big(\tilde{w}^2-\mathcal{G}w^2\Big)^2\Big]+\\
&+c^2\Big(\tilde{w}^3-w^3\Big)^2+f^2\Big(\tilde{w}^3+\mathcal{G}_3w^3\Big)^2+dt^2\,.
\end{aligned}
\tag{5.21}
$$

It is clear that, in order to obtain in our eight-dimensional supergravity approach a metric such as the one written in eq. (5.21), one must take $h_1 = h_2$, $\lambda_1 = \lambda_2 = -\lambda_3/2 = \lambda$ and $G_1 = G_2$ in our general formalism. Then, it is an easy exercise to find the gauged supergravity variables in terms of the functions appearing in the ansatz (5.21). One has:

$$
\begin{aligned}
e^{\phi} &= \frac{1}{2\sqrt{2}}\Big(a^2+b^2\Big)^{\frac12}\Big(f^2+c^2\Big)^{\frac14}\,,\\
e^{\lambda} &= \Big(a^2+b^2\Big)^{\frac16}\Big(f^2+c^2\Big)^{-\frac16}\,,\\
e^{h_1} &= 2\sqrt{2}ab\mathcal{G}\Big(a^2+b^2\Big)^{-\frac13}\Big(f^2+c^2\Big)^{\frac{1}{12}}\,,\\
e^{h_3} &= \sqrt{2}fc(1+\mathcal{G}_3)\Big(a^2+b^2\Big)^{\frac16}\Big(f^2+c^2\Big)^{-\frac{5}{12}}\,,\\
G_1 &= \mathcal{G}\frac{a^2-b^2}{a^2+b^2}\,,\\
G_3 &= \frac{\mathcal{G}_3f^2-c^2}{f^2+c^2}\,.
\end{aligned}
\tag{5.22}
$$

With the parametrization given above, it is not difficult to solve the constraints (3.18). Actually, one of these constraints is trivial, while the other allows to obtain $\mathcal{G}_3$ in terms of the other variables, namely:

$$
\mathcal{G}_3 = \mathcal{G}^2 + \frac{c(a^2-b^2)(1-\mathcal{G}^2)}{2abf}\,. \tag{5.23}
$$

The relation (5.23), with $a \to -a$, is precisely the one obtained in ref. [14]. One can also prove that eq. (5.23) solves eq. (3.21). Actually, the phase α in this case is:

$$
\cos\alpha = \frac{2abc+(b^2-a^2)f}{(a^2+b^2)\sqrt{c^2+f^2}}\,, \qquad \sin\alpha = \frac{2abf+(a^2-b^2)c}{(a^2+b^2)\sqrt{c^2+f^2}}\,. \tag{5.24}
$$

With all these ingredients it is now straightforward, although tedious, to find the first-order equations for the five independent functions of the ansatz (5.21). The result coincides again with the one written in ref. [14], after changing $a \to -a$, and is given by:

$$
\dot{a} = \frac{c^2(b^2-a^2)+[4a^2(b^2-a^2)+c^2(5a^2-b^2)-4abcf]\mathcal{G}^2}{16a^2bc\mathcal{G}^2}\,,
$$

$$\dot{b} = \frac{c^2(a^2-b^2) + [4b^2(a^2-b^2) + c^2(5b^2-a^2) + 4abcf]\mathcal{G}^2}{16ab^2c\mathcal{G}^2}\,,$$
$$\dot{c} = -\frac{c^2 + (c^2-2a^2-2b^2)\mathcal{G}^2}{4ab\mathcal{G}^2},$$
$$\dot{f} = -\frac{(a^2-b^2)\Big[4abf^2\mathcal{G}^2 + c(a^2f-b^2f-4abc)(1-\mathcal{G}^2)\Big]}{16a^3b^3\mathcal{G}^2}\,,$$
$$\dot{\mathcal{G}} = \frac{c(1-\mathcal{G}^2)}{4ab\mathcal{G}}\,. \tag{5.25}$$

Furthermore, the constants p and q are also easily obtained, with the result:

$$p = (a^2-b^2)c\mathcal{G}^2 + 2abf\mathcal{G}_3\mathcal{G}^2\,,$$
$$q = (b^2-a^2)c - 2abf\,, \tag{5.26}$$

while the Hitchin variables are:

$$x_1 = x_2 = -(a^2+b^2)c\mathcal{G}\,, \qquad x_3 = (a^2-b^2)c - 2abf\mathcal{G}_3\,,$$
$$y_1 = y_2 = 2abcf\mathcal{G}(1+\mathcal{G}_3)\,, \qquad y_3 = 4a^2b^2\mathcal{G}^2\,. \tag{5.27}$$

Equations (5.26) and (5.27) are again in agreement with those given in ref. [14], after changing a by $-a$ as before.

6. Summary and conclusions

In this paper we have studied the supersymmetric configurations of eleven dimensional supergravity which are the direct product of Minkowski four dimensional spacetime and a cohomogeneity one seven dimensional manifold of G_2 holonomy with $S^3\times S^3$ principal orbits. These configurations are obtained by uplifting to eleven dimensions some solutions of eight dimensional gauged supergravity which preserve four supersymmetries and satisfy a system of first-order BPS equations. They can be interpreted as being originated by D6-branes wrapping a supersymmetric three cycle which corresponds to a domain wall in eight dimensional gauged supergravity.

The supersymmetry of the solutions is guaranteed by the BPS equations which, once a careful adjustment of the spin connection and the SU(2) gauge field of the eight-dimensional theory has been made, are the conditions required to have Killing spinors. This adjustment is what is known as the topological twist and is directly related to the projection conditions imposed to the Killing spinors. In this paper we have shown how to generalize this projection with respect to the one used up to now. This generalization amounts to the introduction of a phase α in the radial projection of the Killing spinor and, correspondingly, the twist is implemented by a non-abelian gauge field which is not fixed a priori (as in the previous approaches in the literature) but determined by a first-order differential equation. This gauge field encodes the non trivial fibering of the two three spheres in the special holonomy manifold, while the corresponding radial projection determines the wrapping of

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the D6-brane in the supersymmetric three cycle. Actually we have seen that, for non-zero α, the three cycle on which the D6-brane is wrapped has components along the two S^3's (see eq. (2.33)).

A careful analysis of the conditions imposed by supersymmetry has revealed us that, for a general ansatz as in eqs. (3.1)–(3.3), some algebraic constraints have to be imposed to the functions of the ansatz. We have verified that all metrics studied in the literature are particular solutions of our constraints and, in fact, we have found a map between our system and the one introduced by Hitchin. In particular we have demonstrated that, contrary to the generalized believe, the metrics with $q \neq 0$ can be obtained within the 8d gauged supergravity approach. Our formalism is general and systematic and does not assume any particular form of the seven-dimensional metric.

There are other instances on which the kind of generalized twist introduced here can also be studied, the most obvious of them being the cases of D6-branes wrapping two and four cycles. In the former situation we would have to deal with Calabi-Yau manifolds, whereas when the D6-branes wrap a four cycle the special holonomy manifold would be eight dimensional and would have Spin(7) or SU(4) holonomy depending on whether the cycle is coassociative or Kähler. This would be a powerful technique to seek for complete metrics for these special holonomy manifolds. It would be also interesting to analyze the ten dimensional supergravity solutions which correspond to fivebranes wrapping two and three cycles. The relevant gauged supergravity for these cases lives in seven dimensions. Actually, an implementation of the twisting similar to the one introduced here was used in [27] to obtain the Maldacena-Núñez solution [23] for the supergravity dual of $\mathcal{N} = 1$ super Yang-Mills theory.

It would be interesting to study the effect of turning on fluxes in this framework, extending previous results in refs. [30, 32]. The generalization of the twisting seems general enough so as to deserve a more careful study in many lower dimensional gauged supergravities. In particular, it would be interesting to seek for more solutions that do not correspond to the near horizon limit of wrapped D-branes. It is intriguing, for example, to see whether or not the full flat D-brane solution (i.e. without the near horizon limit being taken on it, such as, for example, the Taub-NUT metric for the D6-brane) can be obtained within lower dimensional gauged supergravity.

We are currently working on these issues and we hope to report on them in a near future.

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A. D=8 gauged supergravity

The maximal gauged supergravity in eight dimensions was obtained in [34] by means of a Scherk-Schwarz [36] reduction of eleven dimensional supergravity on a SU(2) group manifold. In the bosonic sector the field content of this theory includes the metric $g_{\mu\nu}$, a dilatonic scalar ϕ, five scalars parametrized by a 3×3 unimodular matrix L_i^{α} which takes values in the coset $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ and a SU(2) gauge potential A_{μ}^{i}. In the fermionic sector there are two pseudo Majorana spinors ψ_{μ} (the gravitino) and χ_i (the dilatino). The kinetic energy of the coset scalars L_i^{α} is given in terms of the symmetric traceless matrix $P_{\mu ij}$, defined through the expression:

$$P_{\mu ij} + Q_{\mu ij} = L_i^{\alpha}(\partial_{\mu}\delta_{\alpha}^{\beta} - \epsilon_{\alpha\beta\gamma}A_{\mu}^{\gamma})L_{\beta j}\,, \tag{A.1}$$

where $Q_{\mu ij}$ is defined as the antisymmetric part of the right-hand side of eq. (A.1). For convenience we are setting in (A.1), and in what follows, the SU(2) coupling constant to one. Moreover, the potential energy of the coset scalars is governed by the so-called T-tensor, T^{ij}, and by its trace T, which are defined as:

$$T^{ij} = L_{\alpha}^{i}L_{\beta}^{j}\delta^{\alpha\beta}\,, \qquad T = \delta_{ij}T^{ij}\,. \tag{A.2}$$

Let $F_{\mu\nu}^{i}$ denote the field strength of the SU(2) gauge potential A_{μ}^{i}. Then, the lagrangian for the bosonic fields listed above is:

$$\begin{aligned}\mathcal{L} = \sqrt{-g_{(8)}}\Big[&\frac{1}{4}R - \frac{1}{4}e^{2\phi}F_{\mu\nu}^{i}F^{\mu\nu i} - \frac{1}{4}P_{\mu ij}P^{\mu ij} - \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \\ &- \frac{1}{16}e^{-2\phi}\left(T_{ij}T^{ij} - \frac{1}{2}T^2\right)\Big]\,.\end{aligned} \tag{A.3}$$

For any solution of the equations of motion derived from (A.3), one can write an eleven dimensional metric which solves the equations of $D=11$ supergravity. The corresponding uplifting formula is:

$$ds_{11}^2 = e^{-\frac{2}{3}\phi}ds_8^2 + 4e^{\frac{4}{3}\phi}\left(A^i + \frac{1}{2}L^i\right)^2\,, \tag{A.4}$$

where L^i is defined as:

$$L^i = 2\tilde{w}^{\alpha}L_{\alpha}^{i}\,, \tag{A.5}$$

with w^i being left invariant forms of the SU(2) group manifold.

We are interested in bosonic solutions of the equations of motion which are supersymmetric. For this kind of solutions, the supersymmetric variations of the fermionic fields vanish for some Killing spinor ϵ. In general, the fermionic fields transform under supersymmetry as:

$$\begin{aligned}\delta\psi_{\lambda} &= D_{\lambda}\epsilon + \frac{1}{24}e^{\phi}F_{\mu\nu}^{i}\hat{\Gamma}_i(\Gamma_{\lambda}^{\mu\nu} - 10\delta_{\lambda}^{\mu}\Gamma^{\nu})\epsilon - \frac{1}{288}e^{-\phi}\epsilon_{ijk}\hat{\Gamma}^{ijk}\Gamma_{\lambda}T\epsilon\,,\\ \delta\chi_i &= \frac{1}{2}\left(P_{\mu ij} + \frac{2}{3}\delta_{ij}\partial_{\mu}\phi\right)\hat{\Gamma}^{j}\Gamma^{\mu}\epsilon - \frac{1}{4}e^{\phi}F_{\mu\nu i}\Gamma^{\mu\nu}\epsilon - \frac{1}{8}e^{-\phi}\left(T_{ij} - \frac{1}{2}\delta_{ij}T\right)\epsilon^{jkl}\hat{\Gamma}_{kl}\epsilon\,,\end{aligned} \tag{A.6}$$

where the $\hat{\Gamma}_i$'s are the Dirac matrices along the SU(2) group manifold and $D_\mu\epsilon$ is the covariant derivative of the spinor ϵ, given by:

$$D_\mu\epsilon = \Big(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab} + \frac{1}{4}Q_{\mu ij}\hat{\Gamma}^{ij}\Big)\epsilon\,, \tag{A.7}$$

with ω_μ^{ab} being the components of the spin connection.

B. Lagrangian approach to the round metric

In this appendix we are going to derive the first-order equations (2.21) by finding a superpotential for the effective lagrangian $\mathcal{L}_{\rm eff}$ in eight dimensional supergravity. The first step in this approach is to obtain the form of $\mathcal{L}_{\rm eff}$ for the ansatz given in eqs. (2.1) and (2.6). Actually, the expression of $\mathcal{L}_{\rm eff}$ can be obtained by substituting (2.1) and (2.6) into the lagrangian given by eq. (A.3). Indeed, one can check that the equations of motion of eight dimensional supergravity can be derived from the following effective lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm eff} \,=\, e^{4f+3h}\Big[&2(f')^2 + (h')^2 - \frac{1}{3}(\phi')^2 - 4e^{2\phi-2h}(g')^2 + 4f'h' + \\ &+ e^{-2h} + \frac{1}{16}e^{-2\phi} - (4g^2-1)^2 e^{2\phi-4h}\Big]\,,\end{aligned} \tag{B.1}$$

together with the zero-energy condition. In the equations obtained from $\mathcal{L}_{\rm eff}$ it is consistent to take $f = \phi/3$, which we will do from now on. Next, let us introduce a new set of functions:

$$a = 2e^{\frac{2\phi}{3}}\,, \qquad b = \frac{1}{2}e^{h-\frac{\phi}{3}}\,, \tag{B.2}$$

and a new variable η, defined as:

$$\frac{dr}{d\eta} = e^{\frac{4\phi}{3}+3h}\,. \tag{B.3}$$

The effective lagrangian in these new variables has the kinetic term:

$$T = \Big(\frac{\dot{a}}{a}\Big)^2 + \Big(\frac{\dot{b}}{b}\Big)^2 + 3\frac{\dot{a}\dot{b}}{ab} - \frac{1}{4}\frac{a^2}{b^2}\big(\dot{g}\big)^2\,, \tag{B.4}$$

where the dot denotes derivative with respect to η. The potential in $\mathcal{L}_{\rm eff}$ is:

$$V = \frac{a^6b^6}{2}\Big[(1-4g^2)^2\frac{a^2}{32b^4} - \frac{1}{2a^2} - \frac{1}{2b^2}\Big]\,. \tag{B.5}$$

The superpotential for $T - V$ in the variables just introduced has been obtained in ref. [13], starting from eleven dimensional supergravity. So, we shall follow here the same steps as in ref. [13] and define $\alpha^1 = \log a$, $\alpha^2 = \log b$ and $\alpha^3 = \log g$. Then, the kinetic energy T can be rewritten as:

$$T = \frac{1}{2}g_{ij}\frac{d\alpha^i}{d\eta}\frac{d\alpha^j}{d\eta}\,, \tag{B.6}$$

where g_{ij} is the matrix:

$$g_{ij} = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & -\frac{a^2}{2b^2} \end{pmatrix} . \tag{B.7}$$

The superpotential W for this system must satisfy:

$$V = -\frac{1}{2} g^{ij} \frac{\partial W}{\partial \alpha^i} \frac{\partial W}{\partial \alpha^j} , \tag{B.8}$$

where g^{ij} is the inverse of g_{ij} and V has been written in eq. (B.5). By using the values of g_{ij} in (B.7), one can write explicitly the relation between V and W as:

$$V = \frac{1}{5} a^2 \left(\frac{\partial W}{\partial a} \right)^2 + \frac{1}{5} b^2 \left(\frac{\partial W}{\partial b} \right)^2 - \frac{3}{5} ab \frac{\partial W}{\partial a} \frac{\partial W}{\partial b} + \frac{b^2}{a^2} \left(\frac{\partial W}{\partial g} \right)^2 . \tag{B.9}$$

Moreover, it is not difficult to verify, following again ref. [13], that W can be taken as:

$$W = \frac{1}{8} a^2 b \sqrt{(a^2(1-2g)^2 + 4b^2)\,(a^2(1+2g)^2 + 4b^2)} . \tag{B.10}$$

The first-order equations associated to the superpotential W are:

$$\frac{d\alpha^i}{d\eta} = g^{ij} \frac{\partial W}{\partial \alpha^j} . \tag{B.11}$$

By substituting the expressions of W and g^{ij} on the right-hand side of eq. (B.11), and by writing the result in terms of the variables used in section 2, one can check that the system (B.11) is the same as that written in eq. (2.21).

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